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A. V. Kolchin, **B. F. Bezrodnyy**, **M. A. Leeva** (Moscow, MADI). **Some aspects of evolution of the generalised allocation scheme.**

In memory of Valentin Kolchin

Combinatorial problems and methods form a considerable part of the investigations in probability theory. In many works in this area, several directions can be distinguished: combinatorial problems in the theory of stochastic processes; problems related to random mappings and random graphs; and problems on allocating particles into cells.

Recently, *the probabilistic approach* is finding ever increasing application in a wide range of such combinatorial problems [1, 3]. In probabilistic combinatorics, the approach based on the *generalised allocation scheme* has received acceptance; this approach reduces a number of combinatorial problems to problems on sums of independent random variables, the classical subject of investigation in probability theory. The generalised allocation scheme was introduced in [2] and has occupied a prominent place in investigations of asymptotic behaviour of combinatorial objects. This scheme generalises the classical scheme of allocation of particles to cells [3] and hence gets its name.

It is demonstrated in [12] that the generalised scheme of allocating a group of particles and other analogues of the generalised allocation scheme are special cases of the generalised allocation scheme with a random number of particles.

Nowadays, deep investigations of the asymptotic behaviour of various kinds of combinatorial objects with the use of the generalised allocation scheme are carried out by many respected mathematicians, among them Yu. L. Pavlov at the Karelian Scientific Centre of the Russian Academy of Sciences [7], A. N. Chuprunov at the Kazan Federal University [12], and I. Fazekas at the University of Debrecen [13].

In the generalised scheme of allocation of particles, the distribution of the cell contents is represented as a conditional distribution of *independent* random variables under the condition that their sum takes a fixed value. Let $\eta_1, \ldots, \eta_N$ be non-negative integer-valued random variables thought of as some numerical characteristics of a combinatorial structure of n elements constituted by N components such that $\eta_1 + \cdots + \eta_N = n$. If there exist independent random variables $\xi_1, \ldots, \xi_N$ such that the joint distribution of $\eta_1, \ldots, \eta_N$ admits the representation

$$\mathbf{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \mathbf{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N \mid \xi_1 + \cdots + \xi_N = n\}, \qquad (1)$$

where $k_1, \ldots, k_N$ are arbitrary integers, we say that $\eta_1, \ldots, \eta_N$ make up the generalised allocation scheme with parameters n and N, and independent random variables $\xi_1, \ldots, \xi_N$. The random variables $\eta_1, \ldots, \eta_N$ are treated as contents of cells.

In view of independence of the random variables $\xi_1, \ldots, \xi_N$, the investigation of many characteristics of the generalised allocation scheme reduces to problems on sums of independent random variables. In the case where the distributions of the summands are identical and fixed (do not depend on the number of summands), one can make use of the well-developed theory of summation of independent random variables. But in most applications of the generalised scheme the need for *local limit theorems in the triangular array scheme* arises.

Usually (see, e. g., [5, 8]), the random variables $\xi_1, \dots, \xi_N$ are disstributed as follows:

$$\mathbf{P}\{\xi_1 = k\} = \frac{b_k\,\theta^k}{k!\,B(\theta)}, \tag{2}$$

where $b_0, b_1, b_2, \dots$ is some sequence of non-negative numbers, $B(\theta) = \sum_{k=0}^{\infty} b_k\theta^k/k!$, and θ is a parameter which takes positive values in the domain of convergence of the series $B(\theta)$.

The scheme where either

Condition A_r, $r \geqslant 2$**:** $b_0 > 0$, $b_1 = \dots = b_{r-1} = 0$, $b_r > 0$, and the maximum span of distribution (2) is 1; or

Condition A_1**:** $b_0, b_1 > 0$,

is fulfilled, occurs widely and obviously embeds into the general scheme. It is in close relation with the so-called *canonical* one.

In the *classical* scheme of equiprobable allocation of particles into cells

$$\mathbf{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \frac{n!}{k_1! \cdots k_N!\, N^n},$$

and it also is a special case of general scheme (1)–(2).

If relation (1) holds for a particular θ, then it remains true for all positive θ in the domain of convergence of the series $B(\theta)$ (see, e.g., [5]). For a complete description of characteristics of the generalised scheme of allocation, one, as a rule, needs local limit theorems for all values of the parameter θ. Major domains of variation of N and θ were analysed in [8, 9, 10, 11].

The objective is to give limit theorems for sums of the form $S_N = \sum_{k=1}^{N} \xi_k$ as $N \to \infty$ while the behaviour of the parameter $\theta = \theta(N)$ varies. Let $P_N(n) = \mathbf{P}\{S_N = n\}$.

In the *most simple case* [8, 11], the values of θ are separated from 0 and do not approach the convergence radius R of the series $B(\theta)$, the following local limit theorem is true.

Theorem. *Let Condition A_r, $r \geqslant 1$, be satisfied; let $N \to \infty$ and let there exist positive constants θ_0, θ_1, such that the parameter $\theta = \theta(N)$ varies in such a way that $\theta_0 \leqslant \theta \leqslant \theta_1 < R$. Then*

$$\sigma\sqrt{N}\, P_N(n) - \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} \to 0$$

uniformly in non-negative integers n, where

$$z = \frac{n - Nm}{\sigma\sqrt{N}}\,.$$

As $N \to \infty$ and $\theta \to 0$, the phenomenon of *transition of distribution of S_N from one lattice to another* manifests itself [9, 10]. In the classical scheme of allocating particles into cells this effect was first observed in [4] for the distribution of the number of cells containing precisely one particle. Namely, convergence to the normal law occurs on the lattice of integers, while convergence to the Poisson law takes place on the lattice of non-negative integers with span r. In more detail, if $N\theta^s \to \infty$ as $N \to \infty$, then S_N obeys the local limit theorem on convergence to the normal law on the lattice of integers. But if

$$N\theta^r \to \lambda\, r!\,\frac{b_0}{b_r}, \qquad \lambda > 0,$$

then S_N obeys the theorem on convergence to the Poissson law with parameter λ on the lattice of non-negative integers with span r. Thus, for intermediate rates of convergence of the parameter θ to zero, the distribution of S_N is subject to transition from the lattice of all non-negative integers to the lattice of non-negative integers with span r.

In the important case where the values of the parameter θ approach the boundary of the domain of convergence of the series $B(\theta)$, other limit laws may arise.

REFERENCES

1. *Goncharov V. L.* On the field of combinatory analysis. — Amer. Math. Soc. Transl., 1962, v. 19, p. 1–46.
2. *Kolchin V. F.* A class of limit theorems for conditional distributions. — Liet. Mat. Rink., 1968, v. 8, p. 53–63.
3. *Kolchin V. F.*, *Chistyakov V. P.,* Combinatorial problems of probability theory. — J. Soviet Math., 1975, v. 4, p. 217–243.
4. *Kolchin V. F.*, *Sevastyanov B. A.*, *Chistyakov V. P.* Random Allocations. N.Y.: Winston, 1978.
5. *Kolchin V. F.* Random Mappings. N.Y.: Optimization Software, 1986.
6. *Kolchin V. F.* Random Graphs. Cambridge: Cambridge Univ. Press, 1999.
7. *Pavlov Yu. L.* Random Forests. Utrecht: VSP, 2000.
8. *Kolchin A. V.* On limit theorems for the generalised allocation scheme. — Discrete Math. Appl., 2003, v. 13, p.627–636.
9. *Kolchin A. V.*, *Kolchin V. F.* On transition of distributions of sums of independent identically distributed random variables from one lattice to another in the generalised allocation scheme. — Discrete Math. Appl., 2006, v. 16, p. 527–540.
10. *Kolchin A. V.*, *Kolchin V. F.* On the transition of distributions of sums of random variables related to the generalised allocation scheme from one lattice to another. — Discrete Math. Appl., 2007, v. 17, p. 455–461.
11. *Kolchin A. V.*, Limit theorems in the generalised allocation scheme. — OP&PM Surveys Appl. Industr. Math., 2009, v. 16, p. 432–435 (in Russian).
12. *Chuprunov A. N.*, *Fazekas I.* On the generalised allocation scheme with a random number of particles. — Discrete Math. Appl., 2012, v. 22, p. 235–240.
13. *Fazekas I.*, *Porvázsnyik B.* A generalized allocation scheme. — Ann. Math. Inform., 2012, v. 39, pp. 57–70.