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"SEVENTH CHORD" OF METHODS FOR SEARCHING CHARACTERIZING MOMENT IDENTITIES AND THE FIFTH ELEMENT
(DEDICATED TO 50TH ANNIVERSARY OF STEIN'S METHOD)

Abstract. A new well-founded method of obtaining Stein-like moment characterizations of one-dimensional probability distributions is suggested. The method is based on functional-operator approach.

Keywords: annihilator of probability measure, generating operator, moment operator, orthogonal polynomial, characterizator of the probability measure, characterizing identity, Stein – Chen-like characterization.

§ 1. Introduction

Interest in the problems of moment characterizations of probability distributions does not fade due to the success of Stein's method, which has become, in particular, one of the important tools for obtaining bounds on errors in approximation problems for sums of *dependent* random variables. Moreover, in the cases important in practice this method often makes it possible to estimate the distances between pre-limit and limit (called target or nested as well) distributions and to compute sharp rates of convergence in stochastic approximation problems, or to compare pairs of distributions (see, for example, (Ross, 2011; Barbour and Chen, 1972) and Ley et al. (2017).

This year we mark the fiftieth anniversary of publication of the seminal Charles Stein's classic paper (Stein, 1972) which turned out to be the source of a fruitful branch of contemporary probability theory. On this occasion, we would like to draw attention to essential contributions to this area regarding the paradigm of Stein's method and to make some comments on interesting details which appears to be missed by experts in the Stein's method and its users.

§ 2. Linear characterizing identities for probability distributions in the operator setting

The original Stein's (or Stein–Chen) method is based on two moment identities valid for random variables having Gaussian or Poisson distributions correspondingly. These identities are exceedingly valuable owing to their so-called *characterizing property*: if for a random variable the identity is valid on some special class of functions then the only possible distribution of this variable is Gaussian or Poisson correspondingly (see (Stein, 1972; Chen, 1975).

for the hypergeometric distribution with parameters n, N, R (cf. (Holmes, 2004, p. 55)), a little bit different coefficient at $\mathcal{X}$ term is obtained in (Schutens, 2001, subsection 5.8),

$$\mathcal{A}_{t(n)} = \mathcal{X}^2\mathcal{D} - (n-1)\,\mathcal{X} + n\mathcal{D} \tag{23}$$

for the Student's t-distribution t_n with $n = 1, 2, \ldots$ degrees of freedom (cf. (Schoutens, 2001, subsection 5.4)). All distributions characterizing by operators (20)–(23) are members of Pearson's family in the continuous case or Ord's family in the discrete case (Schoutens, 2001; Afendras et al., 2018) In some sense the characterizing identity for the circular law with radius 2

$$\mathcal{A}_{\mathrm{Semicircle}(2)} = \mathcal{X}^2\mathcal{D} + 3\,\mathcal{X} - 4\,\mathcal{D}$$

(cf. (Götze and Tikhomirov, 2006, eq. 3.1)) is out of line.

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